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Further Remarks on Linear Groups

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INTRODUCTION

In this paper, we will continue a theme begun at the end of the paper [9]. That is, we will investigate, via algebraic methods, restrictions on the eigenvalues of elements (in finite subgroups of $GL(n, \mathbb{C})$) which will ensure that those elements commute with all their conjugates, a problem with its origins in the work of Blichfeldt and Frobenius. While Blichfeldt's bound as given in Theorem C of [9] is optimal in a sense, Theorems D, D′, and E of [9] show that for primes $p \neq 5$, that bound can be substantially improved for elements of p-power order.

Here, we show that, except possibly for 5-singular elements, the above-mentioned bound can be improved. We hope that the methods which allow us to reduce to the case of elements of prime-power order will be of interest. In fact, we are able to reduce the question in essence to one about elements of p-power order in linear groups G such that $F^*(G)$ is quasi-simple and $G/F^*(G)$ is a cyclic p-group. Our results are set up in such a way that if optimal bounds for such elements in these groups were ascertained, all bounds given here could be improved.

In the final section, we apply J. G. Thompson's theory of quadratic pairs to prove that the obstacle to improving Blichfeldt's bound for 5-singular elements can be removed in linear groups which do not have any composition factor isomorphic to A_5.

1. ON REDUCTION (mod p)

From now on, unless otherwise stated, G is a finite subgroup of $GL(n, \mathbb{C})$. Let p be a prime divisor of the order of G, $\omega = \exp[2\pi i/|G|]$, ρ

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be a prime ideal of $\mathbb{Z}[\omega]$ containing p, R_ρ denote the localization of $\mathbb{Z}[\omega]$ at ρ, K denote $\mathbb{Q}[\omega]$, and let F_ρ denote $R_\rho/J(R_\rho)$. As is well known G is conjugate within $GL(n, \mathbb{C})$ to a subgroup of $GL(n, R_\rho)$. This gives rise to a homomorphism $\sigma_p: G \to GL(n, F_\rho)$. It is always the case that $\ker \sigma_p$ is a (normal) p-subgroup of G. We will refer to the group $G\sigma_p$ as a *reduction* (mod p) of G. We let V denote the associated (R_ρ-free) $R_\rho G$-module, and $\bar{V}$ denote the associated $F_\rho G$-module (so $\bar{V} = V/J(R_\rho)V$). We let V' denote the KG-module $V \otimes_{R_\rho} K$, χ denote the character of G afforded by V', and for $g \in G$, we let $\bar{g}$ denote $g\sigma_p$. As usual, $\pi(G)$ denotes the set of prime divisors of the order of G.

LEMMA 1.1. *Let a be a p-element of G and B be a p'-subgroup of $C_G(a)$. Suppose that in some reduction* (mod p) *of G, $\bar{a}$ acts with minimum polynomial of degree k. Then there is some (necessarily $\langle a \rangle$-invariant) homogeneous component of* $\operatorname{Res}^G_B(V')$ *on which a has at least k distinct eigenvalues.*

Proof. Since $|B|$ is invertible in R_ρ and K contains a splitting field for B, we may write

$$\operatorname{Res}^G_B(V) = \bigoplus Ve_\mu, \text{ where } e_\mu = |B|^{-1} \sum_{b \in B} \mu(1_G)\,\mu(b^{-1})b,$$

and μ runs over irreducible characters of B such that $(\operatorname{Res}^G_B(\chi), \mu) \neq 0$. Denote Ve_μ by W_μ. Then each W_μ is $\langle a \rangle$-invariant, and each W'_μ is homogeneous as KB-module. There must be some μ for which $\bar{W}_\mu(1-\bar{a})^{k-1} \neq \{0\}$. The minimum polynomial of a on that W'_μ is of the form

$$\prod_{i=1}^{t} (X - \alpha_i), \text{ where the } \alpha_i\text{'s are distinct } p\text{-power}$$

roots of unity (in fact, all the distinct eigenvalues of a on W'_μ), so that $\bar{W}_\mu(1-\bar{a})^t = 0$, and hence $t \geqq k$.

Remark. We will be interested primarily in the case that B is cyclic, in which case the W'_μ are just eigenspaces of a generator of B.

2. ELEMENTS WITH CRAMPED EIGENVALUES

DEFINITIONS. Let σ be a finite set of rational primes. We define $\operatorname{ph}(\sigma)$ (the phase of σ) to be $\inf\{\theta$: there is a finite subgroup, H say, of $GL(n, \mathbb{C})$, containing a σ-element, x say, which does not commute with all its H-conjugates, but whose eigenvalues all lie on an arc of length θ on $S^1\}$.

We say that an element, x, of finite order in $GL(n, \mathbb{C})$ has *cramped* eigenvalues if the eigenvalues of x all lie on an arc of length less than $\min(\{\pi\} \cup \{\mathrm{ph}(p), \pi(p-1)/p : p \in \pi(\langle x \rangle) - \{2\}\})$ on S^1.

Remark. It is immediate from the definition that $\mathrm{ph}(\sigma) \leqq \min\{\mathrm{ph}(p) : p \in \sigma\}$ for any set of primes σ. Our aim for the moment is to show that $\mathrm{ph}(\sigma)$ does not become too much smaller than $\min\{\mathrm{ph}(p) : p \in \sigma\}$. We also recall (from Theorems D, D', and E of [9]) that when p is a prime we have:

$\mathrm{ph}(p) = \pi$ if $p = 2$,
$\mathrm{ph}(p) = 2\pi/p$ if $p = 3$ or 5,
$\mathrm{ph}(p) \geqq \max\{\pi(p - \varepsilon(p))/2p,\ [\pi(p - \delta(p))/4p] + \pi/3\}$ if $p > 5$ (where $\varepsilon(p) = \pm 1$ and $p \equiv \varepsilon(p) \pmod 4$,
$\delta(p) \in \{1, -1, 3, 5\}$ and $p \equiv \delta(p) \pmod 8$)).

In particular, we note that $\mathrm{ph}(p) > \pi/2$ except for $p = 5$ and (possibly) $p = 13$.

The first result towards our present aim may have some independent interest.

THEOREM 2.1. *Let G be a finite subgroup of $GL(n, \mathbb{C})$, let τ be a finite non-empty set of rational primes, and let x be an element of G whose eigenvalues all lie on an arc of length less than* $\min(\{\pi\} \cup \{\pi(p-1)/p : p$ *is an odd prime in* $\tau\})$. *Then $O_{\tau'}(C_G(x_\tau)) = O_{\tau'}(G)$ (here, as usual, x_τ denotes the τ-part of x).*

Proof. Let $y = x_\tau$, $z = x_{\tau'}$. We proceed by induction on $|G| + |\tau| + |\langle yZ(G) \rangle|$. We may suppose that G is irreducible. Choose a prime p in τ and let $\sigma = \tau - \{p\}$. Suppose first that $y_p (= x_p)$ is central in G.

If σ is empty then the conclusion of the theorem is certainly valid in the case presently under consideration. If σ is non-empty, then we may assume by induction that $O_{\sigma'}(C_G(y_{p'})) = O_{\sigma'}(G)$ (notice that $x_\sigma = y_{p'}$). Hence y centralizes $O_{\sigma'}(G)$ (as y_p certainly does). We also have $O_{\tau'}(C_G(y)) = O_{\tau'}(C_G(y_{p'})) \leqq O_{\sigma'}(C_G(y_{p'})) = O_{\sigma'}(G)$. Since y centralizes $O_{\sigma'}(G)$, we see that $O_{\tau'}(C_G(y)) \lhd O_{\sigma'}(G)$, so that $O_{\tau'}(C_G(y)) \leqq O_{\tau'}(O_{\sigma'}(G)) = O_{\tau'}(G)$. Since we already know that y centralizes $O_{\tau'}(G)$, we have $O_{\tau'}(C_G(y)) = O_{\tau'}(G)$ in this case.

Hence we may suppose that there is no prime q in τ such that the q-part of y is central in G. Let a be the p-part of y, and let $|\langle aZ(G) \rangle| = p^r$. Then the eigenvalues of a are separated from each other by a distance of at least $2\pi/p^r$ on S^1. It follows from the hypothesis on the eigenvalues of x that on any eigenspace of $x_{p'}$, a has at most $p^{r-1}(p-1)/2$ distinct eigenvalues if p is odd, or at most 2^{r-1} distinct eigenvalues if $p=2$.

By Lemma 1.1, we see that in any reduction (mod p) of G we have:

$$(\bar{a}-1)^{p^{r-1}(p-1)/2} = (\bar{a}^{p^{r-1}}-1)^{(p-1)/2} = 0 \qquad \text{if} \quad p \text{ is odd}$$

$$(\bar{a}-1)^{2^{r-1}} = (\bar{a}^{2^{r-1}}-1) \qquad = 0 \qquad \text{if} \quad p=2.$$

If $p=2$, then $a^{2^{r-1}} \in O_2(G)$, in which case $O_{2'}(G) = O_{2'}(C_G(a^{2^{r-1}}))$ by Lemma 1 of [6]. If p is odd, then $O_{p'}(G) = O_{p'}(C_G(a^{p^{r-1}}))$ by Theorem A of [6]. In either case we have $O_{p'}(G) = O_{p'}(C_G(a^{p^{r-1}}))$. Now $a^{p^{r-1}}$ is not central in G so we may assume by induction that $O_{\tau'}(C_G(y)) = O_{\tau'}(C_G(a^{p^{r-1}}))$. Hence we certainly have $O_{\tau'}(C_G(y)) \leqq O_{p'}(G)$. Since p was an arbitrary prime in τ, we have $O_{\tau'}(C_G(y)) \leqq O_{\tau'}(G)$.

Now if $\tau = \{p\}$ for a single prime p, we already have $O_{\tau'}(C_G(y)) = O_{\tau'}(G)$. Suppose then that $|\tau| > 1$. Then by induction we have $O_{p'}(C_G(x_p)) = O_{p'}(G)$ (using $\{p\}$ in place of τ). In particular, y_p centralizes $O_{p'}(G)$, so that y_p centralizes $O_{\tau'}(G)$. Since p was an arbitrary prime in τ, we have $O_{\tau'}(G) \leqq O_{\tau'}(C_G(y))$. The proof is complete.

LEMMA 2.2. *Let G be a finite subgroup of $GL(n, \mathbb{C})$; let x be an element of G with cramped eigenvalues. Then x commutes with all its conjugates.*

Proof. Let $\sigma = \pi(\langle x \rangle)$. Using Lemma 2 of [9], we may assume that G is irreducible and primitive. By Theorem E of [9], we may assume that $\sigma \neq \{2\}$. Let p be any prime in σ. On any eigenspace of $x_{p'}$, all eigenvalues of x_p lie on an arc of length less than $\mathrm{ph}(p)$ so that x_p commutes with all its $C_G(x_{p'})$-conjugates (and so it lies in $F(C_G(x_{p'}))$ by definition of $\mathrm{ph}(p)$). Hence x_p lies in $O_p(G)$ by Theorem 2 (applied with $\tau = \sigma - \{p\}$).

Since p was arbitrary, x lies in $F(G)$. But as G is irreducible and primitive, $F(G)$ is a central product of $Z(G)$ and (possibly) some extra-special groups. By Lemma 2 of [9] (and the hypotheses on the eigenvalues of x), $\mathrm{trace}(x) \neq 0$, so we must have $x \in Z(G)$, which suffices to complete the proof.

COROLLARY 2.3. *If σ $(\neq \{2\})$ is a finite non-empty set of rational primes, then we have*

$$\mathrm{ph}(\sigma) \geqq \min\{\mathrm{ph}(p),\ \pi(p-1)/p : p \in \sigma - \{2\}\}.$$

In fact, we can sharpen Lemma 2.2 a little:

LEMMA 2.4. *Let G be a finite subgroup of $GL(n, \mathbb{C})$, τ be a finite set of primes, and x be an element of G such that the eigenvalues of x all lie on an arc of length less than* $\min(\{\pi\} \cup \{\pi(p-1)/p : p \in \tau - \{2\}\})$. *Then either*:

(i) *The τ'-part of x commutes with all its G-conjugates or*

(ii) *There is an eigenvalue, say λ, of x_τ such that $x_{\tau'}$ does not have cramped eigenvalues in the representation of $C_G(x_\tau)$ on the λ-eigenspace of x_τ.*

Proof. The given representation of G gives rise to an embedding of $C_G(x_\tau)$ into the direct product $\prod_{\lambda \in \mathrm{Spec}(x_\tau)} GL(n_\lambda, \mathbb{C})$, where n_λ is multiplicity of λ as an eigenvalue of x_τ.

Let σ_λ: $C_G(x_\tau) \to GL(n_\lambda, \mathbb{C})$ be the representation of $C_G(x_\tau)$ afforded by the action of $C_G(x_\tau)$ on the λ-eigenspace of x_τ. Suppose then that $(x_{\tau'})\,\sigma_\lambda$ has cramped eigenvalues for each λ. We prove by induction on $n+|G|$ that $x_{\tau'}$ commutes with all its G-conjugates.

We may suppose that G is irreducible. Next, we claim that we may suppose that G is primitive. Let χ be the character afforded by the given representation of G, and suppose that $\chi = \mathrm{Ind}_H^G(\psi)$ for some primitive irreducible character of some proper subgroup, H, of G. By Lemma 2 of [9],

$x \in \bigcap_{g \in G} H^g$ ($=K$, say). By induction, we may suppose that for any $g \in G$, $x_{\tau'}^g$ commutes with all its H-conjugates (so that $[H, x_{\tau'}^g] \leqq \ker \psi$, as $H/\ker \psi$ is primitive and hence all its Abelian normal subgroups are central). Thus $[K, x_{\tau'}] \leqq \ker \psi^g$ for all g in G, so it easily follows from Clifford's theorem that $x_{\tau'} \in Z(K)$. In that case, all G-conjugates of $x_{\tau'}$ lie in the Abelian normal subgroup $Z(K)$ (of G), so that $x_{\tau'}$ does commute with all its G-conjugates.

We suppose, then, that G is primitive. From Lemma 2.2, we see that for each eigenvalue, λ, of x_τ $(x_{\tau'})\,\sigma_\lambda$ commutes with all its conjugates in the group $C_G(x_\tau)\,\sigma_\lambda$. Thus for any c in $C_G(x_\tau)$, $[x_{\tau'}, x_{\tau'}^c] \leqq \ker \sigma_\lambda$, so $[x_{\tau'}, x_{\tau'}^c] = 1_G$ as λ is arbitrary. Hence $x_{\tau'} \in F(C_G(x_\tau))$.

By Theorem 2.1, we have $O_{\tau'}(C_G(x_\tau)) = O_{\tau'}(G)$, so that $x_{\tau'} \in F(C_G(x_\tau)) \cap O_{\tau'}(G) \leqq F(O_{\tau'}(G)) \leqq F(G)$ (for the first inclusion, note that $O_{\tau'}(G)$ normalizes $F(C_G(x_\tau))$, since we have $O_{\tau'}(G) \leqq C_G(x_\tau)$). Thus $x_{\tau'} \in F(G)$.

At this point, it is as well to note that we have no reason to suppose that $\chi(x_{\tau'}) \neq 0$ for the moment, so that the argument at the end of the proof of Lemma 2.2 cannot be applied.

Let $N = F(O_{\tau'}(G))$, and let η be any irreducible constituent of $\mathrm{Res}^G_{\langle x \rangle N}(\chi)$. Then (as in the proof of Lemma 2 of [9]), $\eta(x) \neq 0$, and it follows that $\mathrm{Res}_N^{N\langle x \rangle}(\eta)$ is irreducible. Since x_τ centralizes N, x_τ is represented by a scalar matrix in any representation of $N\langle x \rangle$ affording η. Now N has the form ZE, where Z is central in G and E is either trivial or a central product of extra-special groups (as G is primitive). If $x_{\tau'}$ does not lie in Z, then $\eta(x_{\tau'}) = 0$ (for again since G is primitive, $\mathrm{Res}_N^G(\chi)$ is a multiple of $\mathrm{Res}_N^{N\langle x \rangle}(\eta)$, and $\chi(x_{\tau'}) = 0$, since any faithful irreducible character of N

vanishes on all non-central elements). But then also $\eta(x)=0$, a contradiction. This contradiction proves that $x_{\tau'} \in Z$, completing the proof of the lemma.

THEOREM 2.5. *Let x, y be elements of finite order in $GL(n, \mathbb{C})$ such that x has cramped eigenvalues and the eigenvalues of y all lie on an arc of length less than π on S^1. Then* either

(i) *x and y commute* or

(ii) *$\langle x, y\rangle$ has an infinite cyclic subgroup.*

Proof. If (ii) does not hold, then $\langle x, y\rangle$ is periodic and finitely generated, so it is finite by a theorem of Schur (36.2 of [1]). We prove by induction on n that x and y commute in that case. We may assume that $\langle x, y\rangle$ is irreducible. By Lemma 2 of [9], $\langle x, y\rangle$ is primitive. By Lemma 2.2, x commutes with all its $\langle x, y\rangle$-conjugates, so that x is central in $\langle x, y\rangle$ as $\langle x, y\rangle$ is primitive, and the proof is complete.

COROLLARY 2.6. *Let G be a finite subgroup of $GL(n, \mathbb{C})$. Then the elements of G which have cramped eigenvalues generate an Abelian normal subgroup of G.*

Remarks. Our next result shows that for larger primes, either $\mathrm{ph}(p) \geqq \pi(p-1)/p$, or else $\mathrm{ph}(p)$ may be computed just by considering subgroups G of $GL(n, \mathbb{C})$ such that $F^*(G)$ is quasi-simple (of order divisible by p) and $G/F^*(G)$ is a cyclic (possibly trivial) p-group (and G satisfies various other restrictive conditions).

THEOREM 2.7. *Let G be a finite subgroup of $GL(n, \mathbb{C})$. Let p be a prime greater than 5 and x be an element of p-power order whose eigenvalues all lie on an arc of length $l < \pi(p-1)/p$ on S^1. Then* either

(i) *$\langle x^g : g \in G\rangle$ is Abelian.* or

(ii) *There is a quasi-simple group Y of order divisible by p which has an irreducible character μ, a proper p-local subgroup W and an automorphism v (possibly inner) of p-power order (which leaves W invariant and stabilizes μ) such that:*

(a) *$\mathrm{Res}^Y_W(\mu)$ is irreducible.*

(b) *W contains every proper $\langle v\rangle$-invariant subgroup of Y.*

(c) *Letting μ' be any extension of μ to $Y\langle v\rangle$, then in any representation of $Y\langle v\rangle$ affording μ', the eigenvalues of (the matrix representing) v all lie on an arc of length at most l on S^1.*

Proof. Choose G, x so that first n, then $|\langle x\rangle|$, then $|G|$ are minimized subject to: the eigenvalues of x all lie on an arc of length at most l on S^1 but x does not commute with all its conjugates. Let χ be the character afforded by the given representation.

Using Lemma 2 and parts of the proof of Theorem D of [9], G is irreducible and primitive, $O_p(G) \leqq Z(G)$, $\Omega_1(\langle x\rangle)$ is not contained in $O_p(G)$, and G is generated by conjugates of x. By Theorem 2.1 (applied with $\tau = \{p\}$), x centralizes $O_{p'}(G)$, so that $O_{p'}(G) \leqq Z(G)$. The choice of G forces $x \in F(H)$ whenever H is a maximal subgroup of G containing x, so by a theorem of Wielandt [11], x lies in a unique maximal subgroup of G, M say. In fact, it easily follows that $M = N_G(X)$, where for some (in fact any) Sylow p-subgroup S of G which contains x, X is the weak closure of $\langle x\rangle$ in S with respect to G. Let

$$C = \left[\bigcup_{m \in M} C_G(x^m)\right] - Z(G).$$

If $c \in C \cap C^g$, say $c \in C_G(x) \cap C_G(x^g)$ (without loss of generality) then $C_G(c) \leqq M$ and x also lies in gMg^{-1}, so that $gMg^{-1} = M$. Since $x \in O_p(M)$, but x lies outside $O_p(G)$, M is not normal in G, so that g lies in M as M is maximal and hence $C^g = C$. Thus C is a T.I. set in M with respect to G.

Now let $xZ(G)$ have order p^r in $G/Z(G)$. Let y be the p^{r-1}th power of x. Then in any reduction (mod p) of G we have $(\bar{y}-1)^{p-1/2} = 0$. Thus by Theorem A of [7], $N_G(Q)$ is irreducible, where Q is the weak closure of $\langle y\rangle$ in S with respect to G. Now $N_G(Q) \leqq M$, so that M is irreducible.

Now $F(G) \leqq Z(G)$, as we have already seen, so that x does not centralise $E(G)$, and hence $G = E(G)\langle x\rangle = G'\langle x\rangle$ (and, in fact, all components of G are $\langle x\rangle$-conjugate by the choice of G).

Now we claim that the given representation of G is tensor indecomposable as a projective representation. For if there were a central extension, H say, of G such that $\chi = \mu\theta$ for non-linear irreducible characters μ and θ of H, then the eigenvalues of x would be all roots of unity of the form $\alpha\beta$ as α runs through all eigenvalues of (a pre-image of) x in any representation of H affording μ and β runs through all eigenvalues of (a pre-image of) x in any representation of H affording θ. In that case, the eigenvalues of this (and hence any) pre-image of x in H in each of these representations would lie on an arc of length at most l on S^1. Since we could choose a pre-image of x in H to be the p-element, say u, u would be represented by a matrix commuting with all its conjugates in each of these representations (and hence be represented by scalar matrices in each case as θ and μ are both primitive characters of H) by the choice of G. But then x would be central in G, contrary to assumption.

Suppose that G has more than one component. Then (for example, by the "Tensor Induction Theorem" of Kovács [4]), the given representation of G is tensor induced (when viewed as a projective representation) from a projective representation of a (normal) subgroup, H, of index p of G. Let ρ be this (projective) representation of H, chosen so that $x^p\rho$ is diagonal. Then we may assume that x is a scalar multiple of $[1 \otimes 1 \otimes 1 \otimes \cdots \otimes x^p\rho] \cdot P$, where P is a permutation matrix of order p (after a suitable choice of coset representatives and replacing G by a conjugate subgroup in $GL(n, \mathbb{C})$ as necessary). In particular, x is a monomial matrix. But now the argument of Lemma 2 of [9] shows that x must be diagonal (if not, there would be a subcharacter ϕ of $\mathrm{Res}^G_{\langle x \rangle}(\chi)$ with $\phi(x) = 0$, in conflict with the hypothesis on the eigenvalues of x), a contradiction.

Thus $G = L\langle x \rangle$, where L is quasi-simple. Also, setting $M_0 = M \cap G'$, we see that M_0 is irreducible (for $M = M_0\langle x \rangle$, $\mathrm{Res}^G_M(\chi)$ is irreducible, and $\chi(x) \neq 0$ by the hypothesis on the eigenvalues of x). Furthermore, M contains every x-invariant proper subgroup of G' (for if U is such a subgroup then $\langle x \rangle U \neq G$ (this is clear if $U \leqq Z(G')$, while if U is non-central, U is not normal in G) so $\langle x \rangle U \leqq M$ and $U \leqq M_0$). We note that M_0 is a p-local subgroup of G', as $M_0 = N_{G'}(X \cap G')$ and $X \cap G'$ is not contained in the centre of G', (otherwise, as G is primitive, $[M, x] \leqq Z(G)$. Since x is not central in G and M is irreducible there is some m in M such that $[x, m] \neq 1_G$. But for that m, we have $\chi(x) = \chi(x^m) = \chi(xz)$ for some non-identity element z of $Z(G)$, and then (as usual) it follows that $\chi(x) = 0$, contrary to the hypotheses on the eigenvalues of x). The result now follows easily.

3. On Reduction (mod 5)

In this section, we use J. G. Thompson's theory of quadratic pairs (see, for example, [10] and Theorem 4.120 of [3]) to investigate the structure of a finite primitive irreducible subgroup G of $GL(n, \mathbb{C})$ which contains a non-identity 5-element which acts with quadratic minimum polynomial in some reduction (mod 5) of G.

Theorem 3.1. *Let G be a finite quasi-primitive irreducible subgroup of $GL(n, \mathbb{C})$ which contains a non-identity 5-element x which acts with quadratic minimum polynomial in some reduction* (mod 5) *of G. Let M be the* (*normal*) *subgroup of G generated by the conjugates of x. Then M is a central product of the form $Z * Y * X$, where Z is a central 5-group, Y is either trivial or extra-special of exponent* 5, *and X is either trivial or a central product of components, each isomorphic to $SL(2, 5)$. If X is non-trivial then $Z(X)$ has order* 2, *and n is even.*

Proof. As in Section 1, we produce an R_5G-module V (affording a representation $\mathbb{C}$-equivalent to the one given) and an F_5G-module $\bar{V}$ on which we assume that x acts with quadratic minimum polynomial. (When necessary, we will extend scalars and view $\bar{V}$ as a kG-module, where k is the algebraic closure of F_5). If $\langle x, y\rangle$ is a 5-group whenever y is a conjugate of x, then M is a 5-group by the Baer–Suzuki theorem, and M has the stated structure (with no components, of course) by [5]. Our proof is complicated by the fact that $\bar{V}$ is unlikely to be absolutely irreducible, and we break it into a series of lemmas.

LEMMA 3.2. *If y is a conjugate of x such that $\langle x, y\rangle$ is not a 5-group, then $\langle x, y\rangle$ is a direct product of a 5-group and a group isomorphic to* $SL(2, 5)$.

Proof. Let $H = \langle x, y\rangle$. By a variant of Blichfeldt's two-eigenvalue argument, the composition factors of $\mathrm{Res}^G_H(\bar{V})$ all have dimension 2 or less. Let $\bar{H} = H/O_5(H)$. Since $\bar{H}$ acts faithfully on the direct sum of the composition factors of $\mathrm{Res}^G_H(\bar{V})$, $\bar{H}$ has Abelian Sylow 5-subgroups. By the Hall–Higman theorem, $\bar{x}$ and $\bar{y}$ both centralize $O_{5'}(\bar{H})$, so that $F(\bar{H}) \leq Z(\bar{H})$. Thus $\bar{H}$ has a component, $\bar{L}$ say (of order divisible by 5), and it follows fairly easily (for example, by a slight modification of the proof of Theorem 2.8.4 of [2]) that $\bar{L} \cong SL(2, 5^r)$ for some r.

We may choose a subnormal subgroup of H, say L, which is minimal subject to covering $\bar{L}$ (and still being subnormal). Then L contains an element, z say, of order 6 whose eigenvalues on any two-dimensional composition factor of $\mathrm{Res}^G_L(\bar{V})$ are the two primitive sixth roots of unity in k, and whose eigenvalue on any one-dimensional composition factor of $\mathrm{Res}^G_L(\bar{V})$ is 1 (for certainly, $L = L'$).

Since z is 5-regular, the eigenvalues of z on $\bar{V}$ "lift" to complex eigenvalues. Since there is at least one two-dimensional composition factor and since by a theorem of Blichfeldt (see Theorem B of [9], for example) z cannot have eigenvalues 1, $-\exp(2\pi i/3)$, $-\exp(4\pi i/3)$, there can be no one-dimensional composition factor of $\mathrm{Res}^G_L(\bar{V})$.

We note now that z^3 is a central involution of G (so that n is even as $z^3 \in G'$) and that z^2 is an element of order 3 of M which has only two eigenvalues in the given representation of G (so that z^2 centralizes any 5-group which it normalizes by the Hall–Higman theorem). Since L is generated by conjugates of z^2, L centralizes any 5-group which it normalizes, so it certainly centralizes $O_5(H)$. Similarly, if p is a prime other than 2 or 5, then whenever u is an element of order p in L, 1 is not an eigenvalue of u in the given representation of G (since this is true of the action of $\bar{u}$ on any composition factor of $\mathrm{Res}^G_L(\bar{V})$), so that u centralizes any 5-group which it normalizes. Now consideration of the Sylow 5-normalizer of $\bar{L}$

forces $5^r - 1$ to be a power of 2, and this in turn implies that $r = 1$, so that $\bar{L} \cong SL(2, 5)$. Since $O_5(L) = O_5(H) \cap L \leqslant Z(L)$, $L = L'$, and $L/O_5(L) \cong SL(2, 5)$, we conclude that $L \cong SL(2, 5)$.

We have seen along the way that each component of $\bar{H}$ has to act faithfully on each composition factor of $\mathrm{Res}_L^G(\bar{V})$ so that $\bar{H}$ has just one component, and hence $\bar{H} \cong SL(2, 5)$, since $\bar{H}$ is generated by elements of 5-power order. Since L is a component of H, it follows that $H = O_5(H) \times L$, completing the proof. We recall for the convenience of the reader that L centralizes $O_5(G)$.

COROLLARY 3.3. *$M^{(\infty)}$ centralizes $F(G)$.*

Proof. By the Hall–Higman theorem, $[O_{5'}(G), x] \leqslant \ker \sigma_5$ (and the same applies to any conjugate of x) so that M centralizes $O_{5'}(G)$, as $\ker \sigma_5$ is a 5-group. Let $K = C_M(O_5(G))$, and let $\bar{M} = M/K$, etc. By Lemma 3.2 (and its proof), $\langle \bar{x}, \bar{x}^{\bar{m}} \rangle$ is a 5-group for every $\bar{m} \in \bar{M}$. By the Baer–Suzuki theorem, $\bar{x} \in O_5(\bar{M})$, so $\bar{M}$ is a 5-group.

LEMMA 3.4. *Every component of M is isomorphic to $SL(2, 5)$.*

Proof. Let L be a component of M. Then we may choose a composition factor, say $\bar{W}$, of $\mathrm{Res}_M^G(\bar{V})$ on which L acts non-trivially. We may suppose that $L \leqq [L, x]$ (for there is some conjugate of x, say y, which does not centralize L and it follows easily from the three subgroups lemma that we must have $L \leqq [L, y]$ for such a y).

By Thompson's results on quadratic pairs, $L/Z(L)$ is a simple group of Lie type in characteristic 5, and letting σ denote the representation of M afforded by W, and $\bar{M}$ denote $M\sigma$, etc., we have $\bar{M} = \bar{L}_1 \bar{L}_2 \cdots \bar{L}_t$ (central product), where each $\bar{L}_i$ is quasisimple (of Lie type in characteristic 5), and each is generated by conjugates of $\bar{x}$. Furthermore, $\bar{W} = \bar{W}_1 \otimes \bar{W}_2 \otimes \cdots \otimes \bar{W}_t$, where each $\bar{W}_i$ is irreducible for $\bar{L}_i$ and each conjugate of $\bar{x}$ lies in precisely one $\bar{L}_i$.

Then $\bar{L} = L\sigma$ is a component of M, and we must have $\bar{x} \in \bar{L}$, as $L \leqq [L, x]$. Then x must normalize L (otherwise $[L, L^x] = \{1_G\}$, whereas $\bar{L}^{\bar{x}} = \bar{L}$ is non-Abelian).

Suppose that $\bar{L}$ does *not* have Lie-rank 1. Then there is certainly more than one maximal parabolic subgroup of $\bar{L}$ which contains $\bar{x}$, so (using [8], for example) there is a maximal 5-local subgroup of $\bar{L}$, say $\bar{P}$, such that $\bar{x}$ does not lie in $O_5(\bar{P})$. We may choose an L-conjugate of x, say $y = x^l$, where $\bar{l} \in \bar{P}$, such that $\langle \bar{x}, \bar{y} \rangle$ is not a 5-group. Then $\langle x, y \rangle^{(\infty)} \cong SL(2, 5)$ as in earlier arguments.

We recall from the proof of Lemma 3.2 that any element of order 3 in $\langle x, y \rangle$ centralizes every $\{2, 3\}'$-subgroup of G which it normalizes. Let v be an element of order 3 in $\langle x, y \rangle$, and let T be the full pre-image of $O_5(\bar{P})$

in L. Then T is certainly nilpotent, and v normalizes T, so that v normalizes (hence centralizes) a Sylow 5-subgroup of T. But then $\bar{v}$ centralizes $O_5(\bar{P})$, a contradiction.

Hence $\bar{L}$ does have Lie rank 1. Now we may choose an $L\langle x\rangle$-conjugate of x, say y, such that $\langle \bar{x}, \bar{y}\rangle$ is not a 5-group. Let u be the unique involution in $\langle x, y\rangle$ (which is central in G as in earlier arguments). Then certainly $\bar{u}$ is central in $\bar{L}$. Since $\bar{L}$ has Lie rank 1, and $Z(\bar{L})$ has even order we must have $\bar{L} \cong SL(2, 5^r)$ for some r. Then $\bar{L}$ is generated by a pair of conjugates of $\bar{x}$, so $\bar{L} \cong SL(2, 5)$ by Lemma 3.2. Thus $L = \langle x, y\rangle^{(\infty)}$ for y above, and we recall from our earlier proof that L acts faithfully on each composition factor of $\mathrm{Res}^G_M(\bar{V})$.

COROLLARY 3.5. *Let $N = M^{(\infty)}$. Then we have $C_N(\bar{W}) \leqq Z(N)$ whenever $\bar{W}$ is a composition factor of* $\mathrm{Res}^G_M(\bar{V})$.

Proof. If not, then we may choose a composition factor $\bar{W}$ so that some component of N (hence of M) acts trivially on $\bar{W}$ (for recall that $F(N) = Z(N)$ by Corollary 3.3).

Conclusion of Proof of Theorem 3.1. Let $\bar{W}$ be a composition factor of $\mathrm{Res}^G_M(\bar{V})$, σ denote the representation of M afforded by W, $\bar{M}$ denote $M\sigma$, etc. As before, we have $\bar{M} = \bar{L}_1\bar{L}_2 \cdots \bar{L}_t$ (a central product), where each $\bar{L}_i$ is quasisimple (of Lie type in characteristic 5).

For $1 \leqslant i \leqslant t$, we choose a subnormal perfect pre-image, L_i, of $\bar{L}_i$ (so each $L_i \leqq N$). We know that N centralizes $F(G)$, so that (for each i), $F(L_i) \leqq F(N) \leqq Z(N)$. But we know that $N \cap \ker \sigma \leqq Z(N)$, and each $L_i/(L_i \cap \ker \sigma)$ is quasi-simple, so it follows that $L_i/F(L_i)$ is simple for each i. Hence each L_i is quasi-simple, thus a component of M, so each is isomorphic to $SL(2, 5)$.

Now $M = E(M) \ker \sigma$, M/M' is a 5-group, as M is generated by conjugates of x. Also, no component of M is contained in $\ker \sigma$, so that $E(M)$ centralizes $\ker \sigma$. Since $E(M)$ acts irreducibly on *every* composition factor of $\mathrm{Res}^G_M(\bar{V})$, it follows that $[M, \ker \sigma] \leqq O_5(M)$ (as $\ker \sigma$ must act as scalars on any such composition factor). Hence $M' \leqq E(M)\, O_5(M)$, so that $M/E(M)$ is a 5-group. Thus $M = QE(M)$, where Q is a Sylow 5-subgroup of $\ker \sigma$, so $M = E(M)\, O_5(M)$ (for Q is certainly normal in M as $E(M)$ centralizes $\ker \sigma$). The structure of $O_5(M)$ has the stated form by [5]. It remains to prove that $Z(E(M))$ has order 2 if $E(M)$ is non-trivial. This follows since (as we have already seen), whenever L is a component of M, the unique involution of L is the unique involution of the centre of G.

THEOREM 3.6. *Let G be a finite irreducible subgroup of $GL(n, \mathbb{C})$, and let x be an element of G which does not commute with all its conjugates.*

Suppose that the eigenvalues of x all lie on an arc of length less than

$$\min(\{4\pi/5\} \cup \{\mathrm{ph}(p),\ \pi(p-1)/p : p \in \pi(x) - \{2, 5\}\}) \qquad \text{on } S^1.$$

Then either x commutes with all its conjugates or else n is even and $\langle x^g : g \in G\rangle$ has a component isomorphic to $SL(2, 5)$.

Proof. We proceed by induction on $|G| + n$. We may assume, then, that if M is any proper subgroup of G which contains x, but has no composition factor isomorphic to A_5, then x commutes with all its M-conjugates.

Suppose that $\langle x^g : g \in G\rangle$ has no such component or that n is odd. We will prove that $\langle x^g : g \in G\rangle$ is Abelian. Let χ be the character afforded by the given representation of G. By the hypotheses on the eigenvalues of x it suffices to prove that x lies in $F(G)$ (for in that case if we assume (as we may) that the elements of $F(G)$ are monomial matrices then all G-conjugates of x are diagonal matrices).

Let y be the 5-part, z be the $5'$-part of x. Then by Lemma 2.4 (applied with $\tau = \{5\}$), z commutes with all its G-conjugates, so that z lies in $F(G)$. We need only prove that y lies in $F(G)$.

We may write $\chi = \mathrm{Ind}_H^G(\psi)$ for some primitive irreducible character ψ of a (not necessarily proper) subgroup, H, of G. Then by Lemma 2 of [9], x lies in every conjugate of H. Let K be the intersection of these conjugates. We may suppose that y does not lie in $F(H)$.

Suppose that y centralizes $F(K)$. Then x does not centralize $E(K)$, so there is a minimal $\langle x\rangle$-invariant product of components of K, say L, with $[L, x] \neq \{1_G\}$. As in several earlier proofs, all irreducible constituents of $\mathrm{Res}^G_{L\langle x\rangle}(\chi)$ restrict irreducibly to L. We may choose such an irreducible constituent μ so that $\mu(y)/\mu(1_G)$ is not a root of unity. This means that $x \ker \mu$ does not commute with all its conjugates in $L\langle x\rangle/\ker \mu$. If $L\langle x\rangle \neq G$, the inductive hypothesis forces us to conclude that $L/\ker \mu$ has a component isomorphic to $SL(2, 5)$, and that $\mu(1_G)$ is even. But then it is easy to see that L must be a central product of groups isomorphic to $SL(2, 5)$, and that $L \leqq \langle x^l : l \in L\rangle$, so the conclusion of the theorem is valid in that case.

If $L\langle x\rangle = G$, then $G = K$, so that $\chi = \psi$, and G is primitive. Since z commutes with all its conjugates, z is now central in G, so we may assume that $x = y$; that is, x is a 5-element. Now $O_5(G) \leqq Z(G)$, as $\mathrm{Res}^G_L(\chi)$ is irreducible. Suppose that $xZ(G)$ has order 5^k in $G/Z(G)$. Then the hypotheses on the eigenvalues of x force x to have at most $2 \times 5^{k-1}$ distinct eigenvalues, and letting w denote the 5^{k-1}-st power of x, it follows that w lies outside $O_5(G)$ and that w acts with quadratic minimum polynomial in any reduction (mod 5) of G. By Theorem 3.1, n is even, and G has a component isomorphic to $SL(2, 5)$. By the minimality of L, this component is contained in $\langle x^g : g \in G \rangle$, so the conclusion of the theorem is valid in this case.

We claim that y centralizes $O_5(K)$. Let $N = O_5(K)$, and let θ be an irreducible constituent of $\mathrm{Res}^H_{N\langle x\rangle}(\psi)$. Then $\mathrm{Res}^{N\langle x\rangle}_N(\theta)$ is irreducible. Since $z \in O_{5'}(G)$, z is represented by a scalar matrix in any representation of $N\langle x\rangle$ affording θ. The hypotheses on the eigenvalues of x imply now that in any such representation, the eigenvalues of the image of y all lie on an arc of length less than $4\pi/5$ on S^1. Since $H/\ker \psi$ is primitive (and we may assume that N is not central in H) we may assume that $N \ker \theta/\ker \theta$ is a central product of an extra-special group and a cyclic group. An argument like that used in step 3 of the proof of Theorem D of [9] shows that $[N, y, N] \leq \ker \theta$. If $[N, y]$ is not contained in $\ker \theta$, then for some n in N, $[n, y]$ is represented by a non-trivial scalar matrix in the corresponding representation, from which it readily follows that $\theta(y) = 0$, contrary to the fact that the eigenvalues of the image of y all lie on an arc of length less than $4\pi/5$ on S^1. Thus $[N, y] \leq \ker \theta$, so in fact $[N, y] \leq \ker \psi$. The same argument may be applied to any G-conjugate of y, so that $[N, y] \leq \ker \psi^g$ for each g in G and y centralizes N.

Now we may suppose that $G = \langle x \rangle\, O_{5'}(F(K))$ (for if not, then the induction hypothesis implies that x lies in the Fitting subgroup of $\langle x \rangle\, O_{5'}(F(K))$ so that y centralizes $O_{5'}(F(K))$ and hence centralizes all of $F(K)$, a case we have dealt with). In particular, we now have $G = H$ and G is primitive. But then z is central, so we may assume that $y = x$. Also, $O_{5'}(F(K))$ is an irreducible subgroup as in earlier arguments. An argument like the one above now shows that some power of y lies outside $O_5(G)$, but acts with quadratic minimum polynomial in any reduction (mod 5) of G, contrary to Theorem 3.1 (or to the Hall–Higman theorem). The proof of Theorem 3.6 is complete.

COROLLARY 3.7. *Let G be a finite subgroup of $GL(n, \mathbb{C})$ with no composition factor isomorphic to A_5, and suppose that x is an element of G whose eigenvalues all lie on an arc of length less than* $6\pi/13$ *on S^1. Then x commutes with all its G-conjugates.*

Proof. This follows from Theorem 3.6, since $\mathrm{ph}(p) \geqq 6\pi/13$ for every prime $p \neq 5$.

Concluding Remarks. It seems quite likely that for all but a small number (probably 1) of primes, we have should have $\mathrm{ph}(p) \geqq \pi(p-1)/p$.

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